1)

a) $\cos(2z)$ is entire, so it is analytic on and interior to C. Since $1+i$ is interior to C, the Cauchy Integral Formula with

$g(z) = \cos(2z)$, $n=2$, $z_0 = 1+i$ is

$$g^{(2)}(1+i) = \frac{2!}{2\pi i} \int_C \frac{g(z)}{(z-(1+i))^3} dz$$

$(\cos(2z))'' = -4\cos(2z)$ so

$$\int_C \frac{\cos(2z)}{(z-1-i)^3} dz = -4\pi i \cos(2+2i)$$

b) $f(z)$ is analytic except at $z=1+i$, which is exterior to C. So f is analytic on and interior to C, and $\int_C f(z)dz = 0$ by Cauchy-Goursat.

2) $1-\frac{1}{z}$ is analytic unless $z=0$, and $\text{Log}(w)$ is analytic unless $w \in \mathbb{R}$, $w \le 0$.

Thus $f(z)$ is analytic unless $z=0$ or

$1-\frac{1}{z} \in \mathbb{R}$, $\quad 1-\frac{1}{z} \le 0$

$\Rightarrow z \in \mathbb{R}$ $\quad \Rightarrow \frac{1}{z} \ge 1$

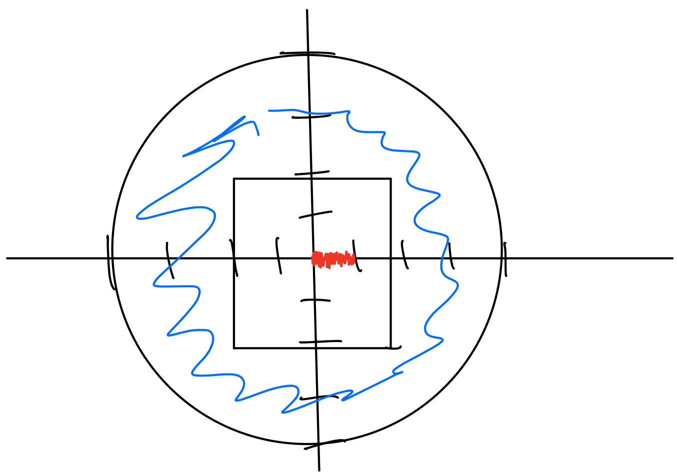
$\Rightarrow z > 0,\ z \le 1$

So $f(z)$ is analytic at all z except $z \in [0,1] \subset \mathbb{R}$.

$[0,1]$ is interior to C_2, which is interior to C_1,

So f is analytic at all points on or between C_1 and C_2 and

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$



3)

a) $$f(z) = \left(\frac{(z-2)+2}{z-2}\right)\frac{1}{-i+(z-2)}$$

$$= \left(1+\frac{2}{z-2}\right)(i)\left(\frac{1}{1-\frac{(z-2)}{i}}\right) \quad \text{Since } \left|\frac{z-2}{i}\right| = |z-2| < 1$$

$$= \left(1+\frac{2}{z-2}\right)(i)\sum_{k=0}^{\infty}\frac{(z-2)^k}{i^k}$$

$$= i\sum_{k=0}^{\infty}\frac{(z-2)^k}{i^k} + \frac{2i}{z-2} + 2\sum_{k=1}^{\infty}\frac{(z-2)^{k-1}}{i^{k-1}}$$

$$= \frac{2i}{z-2} + (2+i)\sum_{k=0}^{\infty}\frac{(z-2)^k}{i^k}$$

b) $$f(z) = \left(1+\frac{2}{z-2}\right)\left(\frac{1}{z-2}\right)\left(\frac{1}{1-\frac{i}{z-2}}\right)$$

Since $\left|\frac{i}{z-2}\right| = \frac{1}{|z-2|} < 1 \quad (|z-2|>1)$

$$= \left(\frac{1}{z-2}+\frac{2}{(z-2)^2}\right)\sum_{k=0}^{\infty}\frac{i^k}{(z-2)^k}$$

$$= \frac{1}{z-2} + \sum_{k=1}^{\infty}\frac{i^k}{(z-2)^{k+1}} + 2\sum_{k=0}^{\infty}\frac{i^k}{(z-2)^{k+2}}$$

$$= \frac{1}{z-2} + i\sum_{k=2}^{\infty} \frac{i^{k-2}}{(z-2)^k} + 2\sum_{k=2}^{\infty} \frac{i^{k-2}}{(z-2)^k}$$

$$= \frac{1}{z-2} + (2+i)\sum_{k=2}^{\infty} \frac{i^{k-2}}{(z-2)^k}.$$

c) Since the series in a converges to f on a deleted neighborhood of $z=2$, $\operatorname{Res}_{z=2} f(z)$ is the coefficient of $\frac{1}{z-2}$, which is $\boxed{2i}$

By Laurent's theorem, the coefficient 1 of $\frac{1}{z-2}$ in the series in b) is $\frac{1}{2\pi i}\int_C f(z)\,dz$ for a circle C centered at 2 with radius >2; since f is analytic outside C, that integral is $-\operatorname{Res}_{z=\infty} f(z)$, so $\operatorname{Res}_{z=\infty} f(z) = -1$.

OR Directly from the definition

$$\operatorname{Res}_{z=\infty} f(z) = -\operatorname{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = -\operatorname{Res}_{z=0} \frac{1/z}{z^2(1/z-2)(1/z-2-i)}$$

$$= -\operatorname{Res}_{z=0} \frac{1}{z(1-2z)(1-2z-iz)}$$

$\phi(z) = \frac{(z-0)}{z(1-2z)(1-2z-iz)} = \frac{1}{(1-2z)(1-2z-iz)}$, $\phi(0)=1$, $\operatorname{Res}_{z=\infty} f(z) = -1.$

4) Singularities at $z = 0, \pm i$ $\quad f(z) = \dfrac{\sin(z)}{z(z+i)(z-i)}$

a)

0

$$\frac{\sin(z)}{z} = \frac{z - \frac{z^3}{6} + \dots}{z} = 1 - \frac{z^2}{6} + \dots \quad \text{on } \mathbb{C} \setminus \{0\}$$

and $\dfrac{1}{(z+i)(z-i)}$ is analytic at $z=0$, so has a taylor series, with nonnegative exponents, on a neighborhood of $z=0$. So $f(z)$ has a series expansion with nonnegative exponents on a deleted neighborhood of $z=0$. So $z=0$ is a removable singularity of $f(z)$.

i

$$\phi_i(z) = (z-i) f(z) = \frac{\sin(z)}{z(z+i)}$$ is analytic and nonzero at $z=i$

So i is a pole of order 1

$-i$

$$\phi_{-i}(z) = (z+i) f(z) = \frac{\sin(z)}{z(z-i)}$$ is analytic and nonzero at $z=-i$

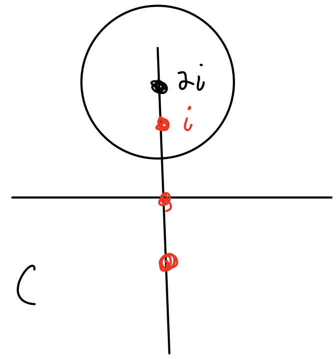
So $-i$ is a pole of order 1

b) $$\operatorname*{Res}_{z=i} f(z) = \phi_i(i) = \frac{\sin(i)}{i(i+i)} = -\frac{\sin(i)}{2} = -\frac{\sinh(1)\, i}{2}$$

$$\operatorname*{Res}_{z=-i} f(z) = \phi_{-i}(-i) = \frac{\sin(-i)}{-i(-i-i)} = -\frac{\sin(-i)}{2} = \frac{\sinh(1)\, i}{2}$$

c) $0, -i$ are exterior to C

i interior to C



f is analytic on and interior to C

except $z = i$

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z) = \pi \sinh(1) = \frac{\pi}{2}\left(e - \frac{1}{e}\right)$$

d) Since f is analytic on and interior to C_R,

$$f'(z_0) = \frac{1}{2\pi i}\int_{C_R} \frac{f(z)}{(z-z_0)^2}\,dz$$

for all z_0 interior to C_R, that is, $|z_0| < R$.

On C_R, $|z| = R$, and $\left|\frac{f(z)}{(z-z_0)^2}\right| = \frac{|f(z)|}{|z-z_0|^2} \le \frac{M}{(|z|-|z_0|)^2} = \frac{M}{(R-|z_0|)^2}$

Since $|z - z_0| \ge \big||z| - |z_0|\big|$.

The length of C_R is $2\pi R$, so the ML estimate gives

$$|f'(z_0)| = \frac{1}{|2\pi i|}\left|\int_{C_R}\frac{f(z)}{(z-z_0)^2}\,dz\right| \le \frac{1}{2\pi}\left(\frac{M}{(R-|z_0|)^2}\right)2\pi R$$

$$= \frac{MR}{(R-|z_0|)^2}.$$